Ministerstwo Nauki i Szkolnictwa Wyższego

Organizacja IX Konferencji Modelowanie Matematyczne w Fizyce i Technice (MMFT 2017)
- zadanie finansowane w ramach umowy 829/P-DUN/2017 ze środków Ministra Nauki i Szkolnictwa Wyższego przeznaczonych na działalność upowszechniającą naukę.

Algebraic dependence of polynomial mappings having two zeros at infinity

Sylwia Lara-Dziembek
Institute of Mathematics, Czestochowa University of Technology

I present explicit formulas that give the algebraic dependence of coordinates of polynomial mapping with the constant Jacobian. Depending on the form of the leading forms of these mappings, we consider two groups of them. Therefore, the formulas indicate, that there are no polynomial which have two zeros at infinity.
Let f_i, h_j be the complex forms of variables X, Y of degrees i, j respectively and $i,\ j \geq 1$.

Remark 1

Let

$$f=(XY)^p+f_{2p-1}+f_{2p-2}+f_{2p-3}+\ldots+f_1 \qquad \text{and} \qquad h=(XY)^q+h_{2q-1}+h_{2q-2}+h_{2q-3}+\ldots+h_1$$

where $p\geq q\geq 1$. *If* $\operatorname{Jac}(f,h)=const=\operatorname{Jac}(f_1,h_1)$ *then* $X^{q-1}Y^{q-1}\big|h_{2q-1}$ *and*

$$f=\left(XY+\frac{1}{q}h_{2q-1|1}\right)^p+A_{p-1}\left(XY+\frac{1}{q}h_{2q-1|1}\right)^{p-1}+\ldots+A_1\left(XY+\frac{1}{q}h_{2q-1|1}\right)$$

$$h=\left(XY+\frac{1}{q}h_{2k-1|1}\right)^q+B_{q-1}\left(XY+\frac{1}{q}h_{2q-1|1}\right)^{q-1}+\ldots+B_1\left(XY+\frac{1}{q}h_{2q-1|1}\right)$$

for some constants $A_1,\ldots,A_{p-1},\ B_1,\ldots,B_{q-1}$. *The form* $h_{2q-1|1}$ *is defined by the formula* $h_{2q-1}=X^{q-1}Y^{q-1}h_{2q-1|1}$.

Remark 2

Let

$$f=\left(X^kY^l\right)^p+f_{(k+l)p-1}+f_{(k+l)p-2}+\ldots+f_{(k+l)(p-1)+1}+\ldots+f_1 \quad \text{and} \quad h=\left(X^kY^l\right)^q+h_{(k+l)q-1}+h_{(k+l)q-2}+\ldots+h_{(k+l)(q-1)+1}+\ldots+h_1$$

where $k>1$ (*k and l are relativity prim*) *and* $p\geq q\geq 1$.

If $\operatorname{Jac}(f,h)=const=\operatorname{Jac}(f_1,h_1)$ *then* $\left(X^kY^l\right)^{q-1}\big|h_{(k+l)q-1}$ *and exist the forms* $\hat{h}_{k+l-2},\ldots,\hat{h}_1$ *for which*

$$f=\left(X^kY^l+\frac{1}{q}h_{(k+l)q-1|k+l-1}+\frac{1}{q}\hat{h}_{k+l-2}+\ldots+\frac{1}{q}\hat{h}_1\right)^p+$$
$$+A_{p-1}\left(X^kY^l+\frac{1}{q}h_{(k+l)q-1|k+l-1}+\frac{1}{q}\hat{h}_{k+l-2}+\ldots+\frac{1}{q}\hat{h}_1\right)^{p-1}+\ldots+$$
$$+A_1\left(X^kY^l+\frac{1}{q}h_{(k+l)q-1|k+l-1}+\frac{1}{q}\hat{h}_{k+l-2}+\ldots+\frac{1}{q}\hat{h}_1\right)$$

and

$$h=\left(X^kY^l+\frac{1}{q}h_{(k+l)q-1|k+l-1}+\frac{1}{q}\hat{h}_{k+l-2}+\ldots+\frac{1}{q}\hat{h}_1\right)^q+$$
$$+B_{q-1}\left(X^kY^l+\frac{1}{q}h_{(k+l)q-1|k+l-1}+\frac{1}{q}\hat{h}_{k+l-2}+\ldots+\frac{1}{q}\hat{h}_1\right)^{q-1}+\ldots+$$
$$+B_1\left(X^kY^l+\frac{1}{q}h_{(k+l)q-1|k+l-1}+\frac{1}{q}\hat{h}_{k+l-2}+\ldots+\frac{1}{q}\hat{h}_1\right)$$

for some constants $A_1,\ldots,A_{p-1},\ B_1,\ldots,B_{q-1}$. *The form* $h_{(k+l)q-1|k+l-1}$ *of degree* $k+l-1$ *is defined by the formula* $h_{(k+l)q-1}=\left(X^kY^l\right)^{q-1}h_{(k+l)q-1|k+l-1}$.

Corollary

In all of these possible cases the polynomials f, h *are algebraically dependent and so* $\operatorname{Jac}(f,h)=0$.

Conclusion

These hypothesis, tested by many examples, allow to state that the polynomial mapping $(f,h):\mathbb{C}^2\to\mathbb{C}^2$ having two zeros at infinity are non-invertible (in the global sense). Therefore it remains to analyze only the case when the mapping (f, h) has only one zero at infinity and thus takes the form

$$f=X^p+f_{p-1}+f_{p-2}+\ldots+f_1 \qquad \text{and} \qquad h=X^q+h_{q-1}+h_{q-2}+\ldots+h_1$$

where f_i, h_j are the forms (of two complex variables) of degrees i, j respectively.

We show that there are non-trivial class of mappings having one zero at infinity with the constant Jacobian, for which that Jacobian vanish. It appears, therefore, that in the general case, the polynomial mapping having one zero at infinity and the constant Jacobian, must be vanish Jacobian. This would mean that the Jacobian Conjecture takes place only in the case, when

$$f=A_0h^n+A_1h^{n-1}+\ldots+A_{n-1}h+a_1X,\quad n\geq 1,\ A_0\neq 0,\ a_1\neq 0 \quad \text{and} \quad h=B_0X^q+B_1X^{q-1}+\ldots+B_{q-2}X^2+h_1,\quad q\geq 2,\ B_0\neq 0,\ \frac{\partial h_1}{\partial Y}\neq 0$$

and also in the simplest case $f=A_0X^n+A_1X^{n-1}+\ldots+A_{n-2}X^2+A_{n-1}X-cY,\quad n\geq 2,\ A_0\neq 0,\ c\neq 0$

and $h=B_0X,\quad B_0\neq 0.$

The polynomials f, h have in each case one zero at infinity and do not have the constant components as well as $\deg f\geq \deg h$.